WHITNEY NUMBER OF CLOSED REAL ALGEBRAIC AFFINE CURVE OF TYPE I

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Abstract. For a closed real algebraic plane affine curve dividing its complexification and equipped with a complex orientation, the Whitney number is expressed in terms of behavior of its complexification at infinity.

1. Introduction

1.1. **Whitney number.** Oriented smooth closed immersed curve C on an oriented affine plane has an important numerical characteristic, *Whitney number,* which is called also *winding number,* and can be defined as the rotation number of the velocity vector, as well as the degree of the Gauss map $C \to S^1$. It determines the immersion $C \to R^2$ up to regular homotopy, i.e. path in the space of immersions.

1.2. **Real Algebraic Curves Under Consideration.** In this paper we consider a class of plane affine real algebraic curves such that the Whitney number is defined naturally for their sets of real points. Namely, we consider irreducible plane affine real algebraic curves A satisfying the following three conditions:

(1) the set of real points $\mathbb{R}A$ is compact,
(2) any real singular point is a non-degenerate double point with real branches,
(3) the set of real points $\mathbb{R}A$ is zero homologous modulo 2 in the set of complex points $\mathbb{C}A \subset \mathbb{C}P^2$ of the projective closure of A.

The second condition implies that $\mathbb{R}A$ can be represented as a smoothly immersed curve. The first condition implies that $\mathbb{R}A$ is closed, i.e., A has no real branches going to infinity. Of course, A has complex branches approaching infinity. They correspond to the intersection points of $\mathbb{C}A$ and $\mathbb{C}P^1_\infty$, the total number of which taken with multiplicities equals the degree of A.

1.3. **Complex Orientations.** The third condition is needed to equip $\mathbb{R}A$ with a natural orientation. Since $\mathbb{R}A$ is zero homologous in $\mathbb{C}A$ modulo 2, it bounds a 2-chain : $\mathbb{R}A = \partial\mathbb{C}A_+ \subset \mathbb{C}A$ modulo 2. Chain $\mathbb{C}A_+$ inherits an orientation from $\mathbb{C}A$ (which has a natural orientation as a complex curve), and induces an orientation on $\mathbb{R}A$. Curve $\mathbb{R}A$ bounds also chain $\mathbb{C}A_- = \operatorname{conj}\mathbb{C}A_+$, where $\operatorname{conj} : \mathbb{C}P^2 \to \mathbb{C}P^2$ is the complex conjugation involution, $\operatorname{conj} : (z_0 : z_1 : z_2) \mapsto (\bar z_0 : \bar z_1 : \bar z_2)$. The

same construction but using $\mathbb{C}A_-$ instead of $\mathbb{C}A_+$, defines the opposite orientation on $\mathbb{R}A$. Both orientations are called *complex.* If $\mathbb{R}A$ has more than one connected components, it has other, non-complex orientations. There are only two chains, $\mathbb{C}A_+$ and $\mathbb{C}A_-$, embedded in $\mathbb{C}A$ and bounded by $\mathbb{R}A$. A choice of a complex orientation is equivalent to the choice between $\mathbb{C}A_+$ and $\mathbb{C}A_-$.

A real algebraic curve A with $\mathbb{R}A$ zero homologous in $\mathbb{C}A$ is said to be of *type I*. This definition is due to Felix Klein. Any real rational curve with infinite $\mathbb{R}A$ is of type I. More about curves of type I and complex orientation can be found in [2], [3] and [4].

For a curve satisfying the conditions of Section 1.2 and equipped with a complex orientation we give an interpretation of the Whitney number $w(\mathbb{R}A)$ in terms of behavior of a half $\mathbb{C}A_+$ of its complexification $\mathbb{C}A$ at infinity.

1.4. **Line at Infinity.** Consider the complex line at infinity $\mathbb{C}P^1_\infty = \mathbb{C}P^2 \smallsetminus \mathbb{C}^2$. Topologically, this is a 2-sphere. The set $\mathbb{R}P^1_\infty = \mathbb{R}P^2 \smallsetminus \mathbb{R}^2$ of its real points is a circle dividing it into two hemi-spheres. These hemi-spheres equipped with the orientations inherited from $\mathbb{C}P^1_\infty$ induce on $\mathbb{R}P^1_\infty$ two orientations opposite to each other. Denote the hemi-sphere which defines the positive (counter-clockwise) orientation on $\mathbb{R}P^1_\infty$ by $\mathbb{C}P^1_{\infty+}$, and the other hemi-sphere by $\mathbb{C}P^1_{\infty-}$.

1.5. **Main Result.** *Let A be a plane affine real algebraic curve satisfying the conditions of Section 1.2 and equipped with the complex orientation defined by $\mathbb{C}A_+$. Then*

$$w(\mathbb{R}A) = \mathbb{C}A_+ \circ \mathbb{C}P^1_{\infty+} - \mathbb{C}A_+ \circ \mathbb{C}P^1_{\infty-}. \tag{1}$$

In the right hand side of (1), $\circ$ means intersection number. 2-Chains $\mathbb{C}A_+$ and $\mathbb{C}P^1_{\infty\pm}$ are compact domains of complex curves in $\mathbb{C}P^2$. Their boundaries $\partial\mathbb{C}A_+$ and $\partial\mathbb{C}P^1_{\infty+} = \partial\mathbb{C}P^1_{\infty-}$ do not meet, there are finitely many points in $\mathbb{C}A_+ \cap \mathbb{C}P^1_{\infty-}$ and $\mathbb{C}A_+ \cap \mathbb{C}P^1_{\infty+}$. Therefore the intersection numbers can be defined as the sums of intersection multiplicities over the intersection points.

1.6. **Reformulation via Asymptotes.** If $\mathbb{C}A$ is transversal to $\mathbb{C}P^1_\infty$ then each point of $\mathbb{C}A \cap \mathbb{C}P^1_\infty$ corresponds to an asymptote of the affine part of $\mathbb{C}A$. If the real affine part of this curve is closed, all the asymptotes are imaginary. Affine imaginary lines which do not meet $\mathbb{R}P^1_\infty$ are divided into those which meet $\mathbb{C}P^1_{\infty+}$ and those which meet $\mathbb{C}P^1_{\infty-}$. Theorem 1.4 claims that $w(\mathbb{R}A)$ equals the difference between the number of the asymptotes of $\mathbb{C}A_+$ of these two sorts.

1.7. **Sketch of Proof and Organization of Paper.** To prove the Main Result, we choose a generic real point on $\mathbb{R}P^1_\infty$ and rotate around it oriented real line L counting changes of $\mathbb{C}A_+ \circ \mathbb{C}L_+ - \mathbb{C}A_+ \circ \mathbb{C}L_-$.

This quantity changes only when $\mathbb{R}L$ kisses $\mathbb{R}A$. The total change can be identified with $-2w(\mathbb{R}A)$ calculated as degree of the Gauss map. On the other hand, at the beginning of rotation, L coincides with P^1_∞, and at the end, with the same line, but with the opposite orientation. Therefore, the total change of the quantity is $-2(\mathbb{C}A_+ \circ \mathbb{C}P^1_{\infty+} - \mathbb{C}A_+ \circ \mathbb{C}P^1_{\infty-})$.

For an expert, this sketch would suffice. To make it more formal, we need to clarify what intersection numbers are to be considered when the real part of the rotating line L would intersect $\mathbb{R}A$. We have to exclude intersection points in the real domain, that is on the boundaries of both 2-chains. To make this in the framework of algebraic topology, we make a spherical blow up of $\mathbb{C}P^2$ along $\mathbb{R}P^2$.

Section 2 is devoted to blow ups of this kind. In Section 3 the changes of the intersection numbers are calculated, and, in Section 4, the proof of Main Result is completed.

2. Digression on Blowing up of Real Point Set

2.1. **Blow up a Submanifold.** For any smooth submanifold Y without boundary of a manifold X one can blow up X along Y in two ways: replacing each point $y \in Y$ either by the projectivization of T_yX/T_yY (i.e., the space of real one-dimensional vector subspaces of T_yX/T_yY), or by the spherization of T_yX/T_yY (i.e., the space of *oriented* one-dimensional vector subspaces of T_yX/T_yY). The first kind of blow up is said to be *projective,* the second one, *spherical.* A projective blow up gives a manifold without boundary, while a spherical one gives a manifold with boundary, which is obtained from Y. If Y is of codimension one in X, the projective blow up does not change X, and the spherical blow up cuts X along Y, i.e., replaces Y by its double covering.

The set of real points $\mathbb{R}A$ of a non-singular real algebraic variety A is a smooth submanifold of middle dimension without boundary of the set $\mathbb{C}A$ of complex points of A. Thus the blow ups outlined above can be made in this situation. The specific of situation provides possibilities for different descriptions of the construction.

Multiplication by $\sqrt{-1}$ defines an isomorphism between $T_y\mathbb{R}A$ and $T_y\mathbb{C}A/T_y\mathbb{R}A$. On the other hand, the projectivization of $T_y\mathbb{R}A$ can be identified with the set of complex one-dimensional subspaces of $T_y\mathbb{C}A$ invariant under the complex conjugation involution $T_y\mathbb{C}A \to T_y\mathbb{C}A$. Therefore projective blow up of $\mathbb{C}A$ along $\mathbb{R}A$ can be considered as replacement of each point of $\mathbb{R}A$ by the set of all real tangent lines of $\mathbb{R}A$ at the point.

Two orientations of a real line are induced on it as on the boundary of the two halves of its complexifications. Therefore, the spherization of $T_y\mathbb{R}A$ can be identified with the set of halves of complexifications

of the real lines. The spherical blow up of $\mathbb{C}A$ along $\mathbb{R}A$ can be identified with replacement of each point of $\mathbb{R}A$ with the set of halves of complexifications of all real tangent lines of $\mathbb{R}A$ at the point.

2.2. **Spherical Blow up of Real Projective Space in Complex Projective Space.** The set of real oriented lines in n-dimensional projective space is naturally identified with the oriented Grassmann variety $G^+_{2,n-1}(\mathbb{R})$. Each point $x \in G^+_{2,n-1}(\mathbb{R})$ is an oriented 2-dimensional vector subspace of $\mathbb{R}^{n+1}$. Its projectivization Px is a line in $\mathbb{R}P^n$ inheriting orientation from x. Take the set

$$\Upsilon P^n = \{(l,p) \in G^+_{2,n-1}(\mathbb{R}) \times \mathbb{C}P^n \mid p \in \mathbb{C}Pl_+\}$$

where $\mathbb{C}Pl_+$ is a hemisphere in the set of complex points of Pl such that the orientation of Pl is induced on Pl as on boundary of $\mathbb{C}Pl_+$ equipped with its complex orientation. The natural projection

$$\upsilon : \Upsilon P^n \to \mathbb{C}P^n : (l,p) \mapsto p$$

is bijective over the set of imaginary points because through any imaginary point one can draw a unique real line (the one that is determined by the point and its image under the complex conjugation). The set of all real oriented lines passing through $p \in \mathbb{R}P^n$ is homeomorphic to sphere S^{n-1}.

Thus ΥP^n can be considered as $\mathbb{C}P^n$ blown up along $\mathbb{R}P^n$. This is an oriented $2n$-dimensional manifold with boundary. The interior of ΥP^n is mapped by υ diffeomorphically onto $\mathbb{C}P^n \smallsetminus \mathbb{R}P^n$. The boundary of ΥP^n is mapped by υ onto $\mathbb{R}P^n$. The map $\partial\Upsilon P^n \to \mathbb{R}P^n$ is a fibration with fiber S^{n-1} equivalent to the fibration of unit tangent vectors of $\mathbb{R}P^n$, spherization of the tangent bundle of $\mathbb{R}P^n$.

2.3. **Non-Singular Real Projective Variety.** The construction of the preceding section is extended naturally to any non-singular real algebraic projective variety: for such a variety $A \subset P^n$ put

$$\Upsilon A = \{(l,p) \in \Upsilon P^n \mid p \in \mathbb{C}A, \text{ and, if } p \in \mathbb{R}A, \text{ then } l \subset T_p\mathbb{R}A\}.$$

If A is a non-singular real projective curve, ΥA can be obtained from $\mathbb{C}A$ by cutting along $\mathbb{R}A$. Recall that cutting of a surface along a curve two-sidedly embedded into the surface is a replacement of the curve by two copies of it.

2.4. **Blow up of Real Part in a Singular Curve.** We need the construction of the preceding section only in the case of projective plane and a plane curve. However, the curve is not assumed to be non-singular.

To encompass this more general situation, consider, for a real plane projective curve A, normalization $\nu : \bar{A} \to A$. The set $\mathbb{C}\bar{A}$ of complex points of $\bar{A}$ is a compact non-singular complex algebraic curve. The restriction of conj to $\mathbb{C}A$ lifts to an anti-holomorphic involution. We

will denote it by c. Observe that $\nu^{-1}(\mathbb{R}A) \supset fix(c)$, but there may be points of $\mathbb{C}\bar{A}$ which are mapped by ν to real points of A without being fixed under c. For example, the preimage under ν of any isolated point of $\mathbb{R}A$ consists of such points.

For each point $x \in \mathbb{C}\bar{A}$, the germ of composition

$$\bar{A} \xrightarrow{\nu} A \xrightarrow{\text{in}} \mathbb{C}P^2$$

has a well-defined osculating line. Denote it by $O(x)$. It passes through $\nu(x)$. If $x \in fix(c)$ then $O(x)$ is real. It may happen that $x \notin fix(c)$, but $O(x)$ is real.

Denote by ΥA the subset of $G^+_{2,2}(\mathbb{R}) \times \mathbb{C}\bar{A}$ consisting of pairs (l, x) such that

- $Pl = O(x)$, if $x \in fix(c)$;
- otherwise just $\nu(x) \in \mathbb{C}Pl_+$

There is a natural map $\Upsilon A \to \mathbb{C}A : (l, x) \mapsto \nu(x)$. On the preimage of the set of non-singular imaginary points of A it is bijective, on the preimage of the set of non-singular real points it is 2-1 map.

3. Intersection of Complex Halves of Curve and Lines

3.1. **When Intersection Is Stable.** Let A be a plane projective real algebraic curve such that its set of real points $\mathbb{R}A$ is zero-homologous modulo 2 in $\mathbb{C}A$. Let $\mathbb{C}A_+ \subset \mathbb{C}A$ be a 2-chain with $\partial\mathbb{C}A+ = \mathbb{R}A$. Let L^t, $t \in \mathbb{R}$ be a continuous family of real projective lines. Suppose their real point sets, $\mathbb{R}L^t$, are coherently oriented and the orientations are induced by the complex orientation of hemispheres $\mathbb{C}L^t_+ \subset \mathbb{C}L^t$.

Lemma 1. *The number of common imaginary points of $\mathbb{C}A_+$ and $\mathbb{C}L^t_+$ counted with multiplicities, as a function of t, is locally constant at all but finite number of values of t, for which $\mathbb{R}L^t$ is either tangent to $\mathbb{R}A$ or passes through singular points of $\mathbb{R}A$.*

Proof. Blow up the sets of real points. This gives rise to oriented compact 2-chains ΥA and ΥL^t in ΥP^2 with $\Upsilon A \cap \partial\Upsilon P^2 = \partial\Upsilon A$ and $\Upsilon L^t \cap \partial\Upsilon P^2 = \partial\Upsilon L^t$.

Each of these 2-chains is covered with two ones, obtained from $\mathbb{C}A_+$ and $\mathbb{C}A_-$ in the case of ΥA and $\mathbb{C}L^t_+$, $\mathbb{C}L^t_-$ in the case of ΥL^t. Denote the 2-chain coming from $\mathbb{C}A_+$ and $\mathbb{C}L^t_+$ by ΥA_+ and ΥL^t_+, respectively. As above, $\Upsilon A_+ \cap \partial\Upsilon P^2 = \partial\Upsilon A_+$ and $\Upsilon L^t_+ \cap \partial\Upsilon P^2 = \partial\Upsilon L^t_+$

The intersection of $\operatorname{Int}\Upsilon A_+$ and $\operatorname{Int}\Upsilon L^t_+$ counted with multiplicities equals the number of common imaginary points of $\mathbb{C}A_+$ and $\mathbb{C}L^t_+$ counted with multiplicities.

If $\mathbb{R}L^t$ does not pass through singular points of $\mathbb{R}A$ and is transversal to $\mathbb{R}A$ then $\partial\Upsilon L^t_+$ and $\partial\Upsilon A_+$ are disjoint. Let U be a regular neighborhood of $\partial\Upsilon A_+$ in $\partial\Upsilon P^2$ disjoint from $\partial\Upsilon L^t_+$, and V be the closure of the complement of U in $\partial\Upsilon P^2$. Surfaces ΥA_+ and ΥL^t_+ realize homology classes belonging to $H_2(\Upsilon P^2, U)$ and $H_2(\Upsilon P^2, V)$, respectively.

There is intersection pairing

$$H_2(\Upsilon P^2, U) \times H_2(\Upsilon P^2, V) \to \mathbb{Z}.$$

Therefore $\operatorname{Int} \Upsilon A_+ \circ \operatorname{Int} \Upsilon L^t_+ = \Upsilon A_+ \circ \Upsilon L^t_+$ can be considered as its value on the homology classes realized by ΥA_+ and ΥL^t_+. The homology class realized by ΥL^t_+ does not change under a sufficiently small change of t. □

Lemma 2. *If, under hypothesis of Lemma 1, A satisfies the conditions listed in Section 1.2 then the number of common imaginary points of $\mathbb{C}A_+$ and $\mathbb{C}L^t_+$ counted with multiplicities jumps only at t for which $\mathbb{R}L^t$ is tangent to a branch of $\mathbb{R}A$.*

Proof. According to condition (2) of Section 1.2, any real singular point of A is a non-degenerate double point with real branches. In the blow up of real part, singularities of this sort are resolved. □

3.2. **When Intersection Jumps.** Under the assumptions above, assume that $\mathbb{R}L^{t_0}$ is tangent quadratically to $\mathbb{R}A$ at a (real) point p and transversal to $\mathbb{C}A$ at all other intersection points. Let t_+ and t_- be real numbers close to t_0 such that

- for all t between each of them and t_0 line $\mathbb{R}L_t$ is transversal $\mathbb{R}A$,
- the number of intersection points of $\mathbb{R}L^{t_+}$ and $\mathbb{R}A$ is greater by 2 than the number of intersection points of $\mathbb{R}L^{t_-}$ and $\mathbb{R}A$.

Lemma 3. *If the complex orientations of A and L^{t_0} at p coincide then*

$$\operatorname{Int} \mathbb{C}A_+ \circ \operatorname{Int} \mathbb{C}L^{t_-}_+ - \operatorname{Int} \mathbb{C}A_+ \circ \operatorname{Int} \mathbb{C}L^{t_+}_+ = 1$$

If the complex orientations of A and L^{t_0} are opposite at p then

$$\operatorname{Int} \mathbb{C}A_+ \circ \operatorname{Int} \mathbb{C}L^{t_-}_+ - \operatorname{Int} \mathbb{C}A_+ \circ \operatorname{Int} \mathbb{C}L^{t_+}_+ = 0$$

This a reformulation of a well-known theorem by Fiedler [1]. A sketch of the original proof is presented below. The problem under consideration is local: near the point of tangency the curve and family of lines are standard up to a local diffeomorphism, which extends locally to complex domain. Therefore it suffices to prove the lemma for any curve and a family of lines of this local diffeomorphic type. For example, we can assume that A is a circle and lines L^t are parallel to each other. Intersection $\mathbb{C}A \cap \mathbb{C}L^{t_-}$ consists of two points, complex conjugate to each other. Hemi-sphere $\mathbb{C}A_+$ meets one of the hemi-spheres $\mathbb{C}L^{t_-}_+$ and $\mathbb{C}L^{t_-}_-$ and does not meet the other one.

A small perturbation of $A \cup L^{t_-}$ gives a non-singular cubic curve with two-component real point set. As an M-curve, it is of type I. Its set of complex points is obtained by a small perturbation from the union of two spheres (which are the sets of complex points of A and L^{t_-}) meeting each other at two points. The perturbation replaces a small disk neighborhoods of an intersection point in the spheres with a tube

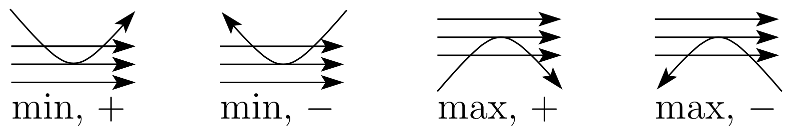

Figure 1.

connecting the boundary circles of the disks. Therefore, $\mathbb{C}A_+$ meets the half of $\mathbb{C}L^{t-}$ which gives together with it a half of the cubic curve.

In the case, when the complex orientations of $\mathbb{R}A$ and $\mathbb{R}L^{t_0}$ at the point of tangency coincide, the orientation of the cubic curve obtained from the complex orientations of $\mathbb{R}A$ and $\mathbb{R}L^{t-}$ coincides with the complex orientation of the cubic curve. See, for example, [2]. Hence, in this case $\mathbb{C}A_+$ meets $\mathbb{C}L^{t-}_+$. $\square$

4. Proof of the Main Result

Choose a direction in which $\mathbb{R}A$ has

- no inflection tangent line,
- no double tangent line,
- no line tangent to a real branch at a singular point.

Let o be the point on $\mathbb{R}P^1_\infty$ which is the common point of all lines of the chosen direction.

Consider the pencil L^t of all real lines passing through o. Orient them coherently. Using L^t, we evaluate sides of (1).

Let $t_1, \dots t_n$ be the values of t for which L^t are tangent to $\mathbb{R}A$. They are divided into four classes, (min, +), (min, −), (max, +), (max, −), according to the change of the number of points in $\mathbb{R}A \cap \mathbb{R}L^t$ when t passes t_i and behavior of orientations of $\mathbb{R}A$ and $\mathbb{R}L^t_i$, see Figure 1.

Obviously,

$$w(\mathbb{R}A) = \#(min,+) - \#(max,+) = \#(max,-) - \#(min,-).$$

Consider now the change of $\mathbb{C}A_+ \circ \mathbb{C}L^t_+ - \mathbb{C}A_+ \circ \mathbb{C}L^t_-$ when t passes the critical value. By Lemma 3, in the case (min,+)

$$\Delta(\mathbb{C}A_+ \circ \mathbb{C}L^t_+) = -1, \quad \Delta(\mathbb{C}A_+ \circ \mathbb{C}L^t_-) = 0, \quad \Delta(\mathbb{C}A_+ \circ \mathbb{C}L^t_+ - \mathbb{C}A_+ \circ \mathbb{C}L^t_-) = -1; \tag{2}$$

in the case (min,−)

$$\Delta(\mathbb{C}A_+ \circ \mathbb{C}L^t_+) = 0, \quad \Delta(\mathbb{C}A_+ \circ \mathbb{C}L^t_-) = -1, \quad \Delta(\mathbb{C}A_+ \circ \mathbb{C}L^t_+ - \mathbb{C}A_+ \circ \mathbb{C}L^t_-) = 1; \tag{3}$$

in the case (max,+)

$$\Delta(\mathbb{C}A_+ \circ \mathbb{C}L^t_+) = 1, \quad \Delta(\mathbb{C}A_+ \circ \mathbb{C}L^t_-) = 0, \quad \Delta(\mathbb{C}A_+ \circ \mathbb{C}L^t_+ - \mathbb{C}A_+ \circ \mathbb{C}L^t_-) = 1 \tag{4}$$

in the case (max,−)

$$\Delta(\mathbb{C}A_+\circ\mathbb{C}L^t_+)=0,\quad \Delta(\mathbb{C}A_+\circ\mathbb{C}L^t_-)=1,\qquad \Delta(\mathbb{C}A_+\circ\mathbb{C}L^t_+-\mathbb{C}A_+\circ\mathbb{C}L^t_-)=-1 \tag{5}$$

Summing up over all critical lines, we obtain that the total change of $\mathbb{C}A_+\circ\mathbb{C}L^t_+-\mathbb{C}A_+\circ\mathbb{C}L^t_-$ is equal to

$$-\#(min,+)+\#(min,-)+\#(max,+)-\#(max,-)=-2w(\mathbb{R}A)$$

On the other hand, the pencil $\mathbb{R}L^t$ starts with $\mathbb{R}P^1_\infty$ and ends up with the same $\mathbb{R}P^1_\infty$, but reverses the orientation. Therefore it exchanges $\mathbb{C}P^1_+$ and $\mathbb{C}P^1_-$. Therefore the total change of $\mathbb{C}A_+\circ\mathbb{C}L^t_+-\mathbb{C}A_+\circ\mathbb{C}L^t_-$ should be equal to $-2(\mathbb{C}A_+\circ\mathbb{C}P^1_{\infty+}-\mathbb{C}A_+\circ\mathbb{C}P^1_{\infty-})$. □

References

[1] T. Fiedler, *Pencils of lines and topology of real algebraic curves*, Izv. Akad. Nauk, Ser. Mat. **46** (1982), 853–863 (Russian), English translation: Math. USSR-Izv. **21** (1983), 161–170.

[2] V. A. Rokhlin, *Complex topological characteristics of real algebraic curves*, Uspekhi Mat. Nauk **33** (1978), 77–89 (Russian).

[3] O. Ya. Viro, *Progress in the topology of real algebraic varieties over the last six years*, Uspekhi Mat. Nauk **41** (1986), 45–67 (Russian), English transl., Russian Math. Surveys **41:3** (1986), 55–82.

[4] V.I.Zvonilov, *Complex orientations of real algebraic curves with singularities*, Soviet Math. Dokl. **27** (1983), 14–17.